

# Which self-maps appear as lattice anti-endomorphisms?

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ABSTRACT. Let  $f : A \rightarrow A$  be a self-map of the set  $A$ . We give a necessary and sufficient condition for the existence of a lattice structure  $(A, \vee, \wedge)$  on  $A$  such that  $f$  becomes a lattice anti-endomorphism with respect to this structure.

## 1. INTRODUCTION

A *partially ordered set* (*poset*) is a set  $P$  together with a reflexive, antisymmetric, and transitive (binary) relation  $r \subseteq P \times P$ . For  $(x, y) \in r$  we write  $x \leq_r y$  or simply  $x \leq y$ . If  $r \subseteq r'$  for the partial orders  $r$  and  $r'$  on  $P$ , then  $r'$  is an *extension* of  $r$ . A map  $p : P \rightarrow P$  is *order-preserving* (*order reversing*) if  $x \leq y$  implies  $p(x) \leq p(y)$  ( $p(y) \leq p(x)$ ) respectively) for all  $x, y \in P$ . The poset  $(P, \leq)$  is a *lattice* if any two elements  $x, y \in P$  have a unique least upper bound (lub)  $x \vee y$  and a unique greatest lower bound (glb)  $x \wedge y$  (in  $P$ ).

In the present paper we consider a self-map  $f : A \rightarrow A$  of a set  $A$ . A list  $x_1, \dots, x_n$  of distinct elements from  $A$  is a *cycle* (of length  $n$ ) with respect to  $f$  if  $f(x_i) = x_{i+1}$  for each  $1 \leq i \leq n-1$  and also  $f(x_n) = x_1$ . A *fixed point* of the function  $f$  is a cycle of length 1, i.e. an element  $x_1 \in A$  with  $f(x_1) = x_1$ . A cycle that is not a fixed point is *proper*.

If  $(A, \vee, \wedge)$  is a lattice (on the set  $A$ ) such that  $f(x \vee y) = f(x) \wedge f(y)$  and  $f(x \wedge y) = f(x) \vee f(y)$  for all  $x, y \in A$ , then  $f$  is a *lattice anti-endomorphism* of  $(A, \vee, \wedge)$ . The square  $f^2 = f \circ f$  of a lattice anti-endomorphism is an ordinary lattice endomorphism of  $(A, \vee, \wedge)$ . A lattice anti-endomorphism is an order-reversing map (with respect to the order relation of the lattice), but the converse is not true in general. More details about lattices and lattice (anti)endomorphisms can be found in [G].

For a proper cycle  $x_1, \dots, x_n \in A$  with respect to a lattice anti-endomorphism  $f$ , if we put

$$p = x_1 \vee x_2 \vee \dots \vee x_n \text{ and } q = x_1 \wedge x_2 \wedge \dots \wedge x_n,$$

then  $p \neq q$ . The equalities

$$\begin{aligned} f(p) &= f(x_1) \wedge f(x_2) \wedge \dots \wedge f(x_n) = x_2 \wedge \dots \wedge x_n \wedge x_1 = q, \\ f(q) &= f(x_1) \vee f(x_2) \vee \dots \vee f(x_n) = x_2 \vee \dots \vee x_n \vee x_1 = p \end{aligned}$$

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show that  $p$  and  $q$  form a cycle of length 2 with respect to  $f$ . If  $u, v \in A$  are distinct fixed points of  $f$ , then  $p = u \vee v$  and  $q = u \wedge v$  form again a cycle of length 2 with respect to  $f$ :

$$f(p) = f(u) \wedge f(v) = u \wedge v = q \text{ and } f(q) = f(u) \vee f(v) = u \vee v = p.$$

It follows that any lattice anti-endomorphism having a proper cycle or having at least two fixed points must have a cycle of length 2.

We prove that the above combinatorial property completely characterizes the possible lattice anti-endomorphisms. More precisely, for a map  $f : A \longrightarrow A$  there exists a lattice  $(A, \vee, \wedge)$  on  $A$  such that  $f$  is a lattice anti-endomorphism of  $(A, \vee, \wedge)$  if and only if  $f$  has a cycle of length 2 or  $f$  has no proper cycles and has at most one fixed point.

In section 2 we give a relatively short and self-contained proof for the mentioned characterization of lattice anti-endomorphisms by using conditional lattices. First we prove that every self-map is a conditional lattice anti-endomorphism with respect to some conditional lattice structure on the base set.

Section 3 is devoted to the study of lattice-order extensions preserving the anti-monotonicity of a given self-map  $f : A \longrightarrow A$ . We show that certain partial orders on  $A$  can be extended to lattice-orders making  $f$  an anti-endomorphism. Thus we obtain a broad generalization of the pure existence result in section 2. Our treatment in section 3 follows the lines of [Sz2], where a combinatorial characterization of ordinary lattice endomorphisms can be found. The results of [FSz] served as a starting point in [Sz2], while in section 3 we build on [Sz1]. The construction in the proof of Theorem 3.8 is based on the use of the maximal antimonicity preserving (AMP) extensions of  $r$  in a partially anti-ordered unary algebra  $(A, f, \leq_r)$ . Such extensions were completely determined in [L] and [Sz1]. In order to make the exposition more self-contained, we present the necessary background about the mentioned AMP extensions.

## 2. CONDITIONAL LATTICES AND ANTI-ENDOMORPHISMS

Recall that a partial order  $(A, \leq_r)$  is called a *conditional lattice* if for any two elements  $x$  and  $y$  the following holds: if the two elements have a common upper bound, then they have a least upper bound, denoted  $x \vee y$ , and if they have a common lower bound, then they have a greatest lower bound, denoted  $x \wedge y$ . In those cases we also say that  $x \vee y$  or  $x \wedge y$  *exist*. All lattices are conditional lattices, so are all antichains, also any poset that is a union of disjoint chains with no comparabilities between elements from the different chains (unordered sum of chains).

An *anti-endomorphism* of a conditional lattice  $(A, \leq_r)$  is a self-map  $f : A \longrightarrow A$  such that for all  $x, y \in A$  the following hold:

$$x \vee y \text{ exists} \iff f(x) \wedge f(y) \text{ exists and equals } f(x \vee y)$$

$$x \wedge y \text{ exists} \iff f(x) \vee f(y) \text{ exists and equals } f(x \wedge y)$$

In what follows we use the language of directed graphs. The *digraph* of a self-map  $f : A \longrightarrow A$  has node set  $A$  and arrow set  $\{(x, f(x)) \mid x \in A\}$ .

**2.1. Conditional Lattice Lemma.** *Every function  $f : A \longrightarrow A$  is an anti-endomorphism of some conditional lattice on  $A$ .*

**Proof.** Without loss of generality assume connectedness, i.e. that the digraph  $D$  on node set  $A$  with arrows  $(x, f(x))$  is simply connected. The *tree components* are the connected components obtained from  $D$  by removing the arrows of any (unique if existing) directed cycle. For any tree component  $T$  the *sons* of a node  $y$  are the nodes  $x$  in  $T$  with  $f(x) = y$ . For each node  $x$  of  $T$  choose a linear order  $\leq_x$  on the set of its sons, having a first son and a last son, if  $x$  has any son. Let  $P_x$  denote the maximal directed path in  $T$  starting from  $x$ . For nodes  $x, y$  in  $T$  let  $z$  be the first common node of  $P_x$  and  $P_y$ , and write  $x \leq_{\text{lex}} y$  whenever (for distance denoted by  $d$ )

$$x = y \text{ or } d(x, z) < d(y, z)$$

or

$$x \neq y, d(x, z) = d(y, z) \text{ and } x' \leq_z y'$$

for the nodes  $x', y'$  preceeding  $z$  immediately in  $P_x, P_y$ .

This defines a linear order on the nodes of  $T$ , call *lexicographic* (lex) order. Define partial orders on  $A$ , distinguishing two cases.

*Case 1:*  $D$  has no cycle. Let  $g : A \rightarrow \mathbb{Z}$  be a grading of  $A$ , i.e.  $g(f(x)) = g(x) + 1$  for all  $x \in A$ . Let  $N(i) = g^{-1}(i)$  for  $i \in \mathbb{Z}$ . Let  $E$  and  $O$  be the set of nodes of even and odd grade, respectively. Order  $E$  lexicographically  $O$  reverse-lexicographically, and let all elements of  $E$  be less than all elements of  $O$ . This is a linear order on  $A$ ,  $f$  is order-reversing.

*Case 2:*  $D$  has a cycle with  $m$  distinct nodes  $a_1, \dots, a_m$  indexed by mod  $m$  integers,  $m \geq 1$ , such that  $f(a_i) = a_{i+1}$ . There is a tree component  $T_i$  for each  $a_i$ . Let  $N(i, k)$  be the set of nodes of  $T_i$  at distance  $k \geq 0$  from  $a_i$ . Order  $N(i, k)$  lex or reverse lex according to whether  $k$  is even or odd.

Let all elements of  $N(i, k)$  be less (greater) than all elements of  $N(i+1, k+1)$  if  $k$  is even (odd), with no further comparabilities.

This order on  $A$  is a conditional lattice,  $f$  is an anti-endomorphism.  $\square$

**2.2. Theorem.** *A function  $f : A \rightarrow A$  is a lattice anti-endomorphism of some lattice structure on  $A$  if and only if either  $f$  transposes some pair of elements or it does not induce a permutation on any finite set of at least two elements.*

**Proof.**

*Case 1:*  $f$  has a single fixed point  $a$  in a connected component  $K$  of its digraph  $D$ , and the other connected components  $K_i$ ,  $i \in \sigma$ , indexed by some ordinal  $\sigma$ , have no cycle. Removing the loop arrow from  $K$  we obtain a tree whose node set can be ordered lexicographically as in the proof of the Lemma. Classify the nodes of  $K$  into even and odd sets  $E, O$  according to the parity of their distance from  $a$ . Classify the nodes of each  $K_i$  into even and odd sets  $E_i, O_i$  according to a  $\mathbb{Z}$ -grading as in the proof of the Lemma.

Let  $E$  and the  $E_i$ 's be ordered lexicographically,  $O$  and the  $O_i$ 's reverse lexicographically. Order  $A$  by

$$\dots < O_2 < O_1 < O_0 < O < E < E_0 < E_1 < E_2 < \dots,$$

i.e. the odd and even sets now appear as succeeding intervals of a linear order on  $A$ .

*Case 2:* If  $f$  has no cycle, not even a fixed point, then simply omit  $K, O, E$  in the above construction.

*Case 3:*  $f$  has a 2-cycle, i.e. for some  $a_1 \neq a_2$ ,  $f(a_1) = a_2$ ,  $f(a_2) = a_1$ . Let  $K$  be the connected component of the digraph of  $f$  containing this 2-cycle, and

let  $B$  denote the elements of  $A$  not in  $K$ . Then both  $B$  and  $A \setminus B$  are closed under  $f$ , and by the above Lemma there is a conditional lattice structure  $L$  on  $B$ , on which  $f$  induces an anti-endomorphism. Let  $T_1, T_2$  be the trees obtained from  $K$  by removing the arrows of the 2-cycle between  $a_1$  and  $a_2$ . For  $i = 1, 2$  and  $k \geq 1$  let  $N(i, k)$  denote the set of nodes of  $T_i$  at distance  $k$  from  $a_i$ . Fix a lexicographic order on the nodes of each  $T_i$  as in the proof of the Lemma. Order  $N(i, k)$  lexicographically for  $k$  even, reverse lexicographically for  $k$  odd. Order  $A$  by

$$\begin{aligned} & \underbrace{\cdots < N(1, 5) < N(1, 3) < N(1, 1)}_{N(1, k) \text{ } k \text{ odd decreasing}} < \underbrace{\cdots < N(2, 5) < N(2, 3) < N(2, 1)}_{N(2, k) \text{ } k \text{ odd decreasing}} < a_1 < L \\ & L < a_2 < \underbrace{N(2, 2) < N(2, 4) < N(2, 6) < \cdots}_{N(2, k) \text{ } k \text{ even increasing}} < \underbrace{N(1, 2) < N(1, 4) < N(1, 6) < \cdots}_{N(1, k) \text{ } k \text{ even increasing}} \end{aligned}$$

i.e. a chain made up by succeeding intervals  $N(i, k)$ ,  $k$  odd, followed by the element  $a_1$ , the conditional lattice  $L$ , then the element  $a_2$ , then by a chain made up by succeeding intervals  $N(i, k)$ ,  $k$  even. This order is a lattice, of which  $f$  is an anti-endomorphism.  $\square$

### 3. LATTICE ORDER EXTENSIONS AND ANTI-ENDOMORPHISMS

The following definitions appear in [FSz] and [Sz1]. The treatment in the paper [JPR] also starts with similar considerations.

Let  $f : A \rightarrow A$  be a function and define the equivalence relation  $\sim_f$  as follows: for  $x, y \in A$  let  $x \sim_f y$  if  $f^k(x) = f^l(y)$  for some integers  $k \geq 0$  and  $l \geq 0$  with  $f^0$  meaning the identity function. The equivalence class  $[x]_f$  of an element  $x \in A$  is called the  $f$ -component of  $x$ . We note that  $f([x]_f) \subseteq [x]_f$ .

An element  $c \in A$  is called *cyclic* with respect to  $f$  (or  $f$ -cyclic), if  $f^m(c) = c$  for some integer  $m \geq 1$ . The *period* of a cyclic element  $c$  is

$$n_f(c) = \min\{m \mid m \geq 1 \text{ and } f^m(c) = c\}.$$

Obviously,  $f^k(c) = f^l(c)$  holds if and only if  $k - l$  is divisible by  $n_f(c)$ . A cyclic element  $c$  or the  $f$ -cycle  $\{c, f(c), \dots, f^{n_f(c)-1}(c)\}$  is called *proper* if  $n_f(c) \geq 2$ . The  $f$ -orbit  $\{x, f(x), \dots, f^k(x), \dots\}$  of  $x$  is finite if and only if  $[x]_f$  contains a cyclic element. If  $c_1, c_2 \in [x]_f$  are cyclic elements, then  $n_f(c_1) = n_f(c_2) = n_f(x)$  and this number is called the *period* of  $x$ . If the  $f$ -orbit of  $x$  is infinite, then define  $n_f(x) = \infty$ . Clearly,  $x \sim_f y$  implies that  $n_f(x) = n_f(y)$ . If  $f$  has a cycle of odd period, then it also defines a cycle of  $f^2 = f \circ f$  with the same elements and the same (odd) period. If  $f$  has a cycle of even period  $n = 2m \geq 2$ , then it is a disjoint union of two  $f^2$ -cycles of period  $m$ . If  $n_f(x)$  is odd, then  $[x]_f = [x]_{f^2} = [f(x)]_{f^2}$  and  $n_{f^2}(x) = n_f(x)$ . If  $n_f(x)$  is even or  $n_f(x) = \infty$ , then  $x \sim_{f^2} f(x)$ ,  $[x]_f = [x]_{f^2} \cup [f(x)]_{f^2}$  and  $n_{f^2}(x) = \frac{1}{2}n_f(x)$ . Notice that  $f([x]_{f^2}) \subseteq [f(x)]_{f^2}$ .

The set of  $f$ -incomparable pairs is  $\pi = \alpha \cup \beta$ , where

$$\alpha = \{(x, y) \in A \times A \mid x \sim_f y, n_f(x) \text{ and } n_f(y) \text{ are odd integers}\},$$

$$\beta = \{(x, y) \in A \times A \mid f^{2k+m}(x) = f^m(y) \neq f^m(x) = f^{2l+m}(y) \text{ for some } k, l, m \geq 0\}.$$

If  $(x, y)$  is not  $f$ -incomparable (i.e.  $(x, y) \notin \pi$ ), then we say that  $(x, y)$  is  $f$ -comparable. The following properties can easily be checked.

1. If  $(x, y)$  is  $f$ -incomparable, then  $(y, x)$  is also  $f$ -incomparable.

2. If  $(f(x), f(y))$  is  $f$ -incomparable, then  $(x, y)$  is also  $f$ -incomparable.
3. If  $f$  has more than one fixed point, then  $\alpha \neq \emptyset$ .
4. If  $(x, y) \in \beta$  and  $m$  is as in the definition of  $\beta$ , then  $f^m(x)$  and  $f^m(y)$  are (different) elements of the same  $f^2$ -cycle.
5. If  $f^2$  has a proper cycle, then  $\beta \neq \emptyset$ .
6. If  $f$  has at most one fixed point and  $f^2$  has no proper cycle, then  $\alpha = \beta = \emptyset$  (hence  $\pi = \emptyset$ ).

**3.1. Lemma.** *For  $x, y \in A$  the following conditions are equivalent*

1.  $(x, y) \in \beta$ ,
2.  $(f(x), f(y)) \in \beta$ ,
3.  $3 \leq n_f(x) \neq \infty$ ,  $x \sim_{f^2} y$  and  $f^t(x) \neq f^t(y)$  for all integers  $t \geq 0$ .

**Proof.** (1) $\implies$ (2)&(3). Now  $f^{2k+m}(x) = f^m(y) \neq f^m(x) = f^{2l+m}(y)$  for some  $k, l, m \geq 0$ . It follows that  $k, l \geq 1$ . Since  $f^{2k+m}(x) = f^m(y)$  implies  $f^{2k+m+1}(x) = f^{m+1}(y)$  and one of  $m$  and  $m+1$  is even, we obtain that  $x \sim_{f^2} y$ . Clearly,  $f^m(y) \neq f^m(x)$  implies that  $f^t(x) \neq f^t(y)$  for all  $0 \leq t \leq m$ . In view of

$$f^{2k+2l-1}(f^{m+1}(x)) = f^{2k+2l+m}(x) = f^{2l}(f^{2k+m}(x)) = f^{2l}(f^m(y)) = f^{2l+m}(y) = f^m(x)$$

and

$$f^{2k+2l-1}(f^{m+1}(y)) = f^{2k+2l+m}(y) = f^{2k}(f^{2l+m}(y)) = f^{2k}(f^m(x)) = f^{2k+m}(x) = f^m(y)$$

the equality  $f^{m+1}(x) = f^{m+1}(y)$  would imply  $f^m(x) = f^m(y)$ , a contradiction. It follows that  $f^{2k+m}(f(x)) = f^m(f(y)) \neq f^m(f(x)) = f^{2l+m}(f(y))$  and  $(f(x), f(y)) \in \beta$ . Starting from  $(f(x), f(y)) \in \beta$  and replacing the triple  $(x, y, m)$  by  $(f(x), f(y), m+1)$  in the above argument give that  $f^{m+2}(x) \neq f^{m+2}(y)$ . Thus  $f^t(x) \neq f^t(y)$  for all  $t \geq m$ . As a consequence of  $f^{2k+2l+m}(x) = f^m(x)$ , we deduce that  $f^m(x)$  is  $f$ -cyclic. Since  $f^{m+2}(x) = f^m(x)$  would imply  $f^m(x) = f^{2k+m}(x) = f^m(y)$ , we obtain that  $3 \leq n_f(x) \neq \infty$ .

(2) $\implies$ (1) is straightforward.

(3) $\implies$ (1).  $x \sim_{f^2} y$  imply the existence of integers  $r, s \geq 0$  such that  $f^{2r}(x) = f^{2s}(y)$ . Since  $3 \leq n_f(x) \neq \infty$ , there exists an integer  $t \geq 0$  such that  $f^{2t+2r}(x) = f^{2t+2s}(y)$  is  $f$ -cyclic of period  $n_f(x)$ . In view of  $2t + 2r \leq 2t + 2r + 2s$  and  $2t + 2s \leq 2t + 2r + 2s$ , the relations

$$f^{2t+2r+2s}(x) \sim_{f^2} x \sim_{f^2} y \sim_{f^2} f^{2t+2r+2s}(y)$$

imply that  $f^{2t+2r+2s}(x)$  and  $f^{2t+2r+2s}(y)$  are different  $f$ -cyclic elements in  $[x]_{f^2}$ . It follows that  $f^{2t+2r+2s}(x)$  and  $f^{2t+2r+2s}(y)$  are in the same  $f^2$ -cycle and

$$f^{2k}(f^{2t+2r+2s}(x)) = f^{2t+2r+2s}(y) \neq f^{2t+2r+2s}(x) = f^{2l}(f^{2t+2r+2s}(y))$$

for some  $k, l \geq 1$  ( $k+l$  is a multiple of  $n_{f^2}(x) = n_{f^2}(y)$ ). Thus  $(x, y) \in \beta$ .  $\square$

**3.2. Corollary.** *For any  $a \in A$  the relation  $([a]_{f^2} \times [a]_{f^2}) \setminus \beta$  is an equivalence relation on the set  $[a]_{f^2}$ . If  $3 \leq n_f(a) \neq \infty$ , then there are exactly  $n_{f^2}(a)$  equivalence classes with respect to  $([a]_{f^2} \times [a]_{f^2}) \setminus \beta$ .*

**Proof.** If  $1 \leq n_f(a) \leq 2$  or  $n_f(a) = \infty$ , then  $([a]_{f^2} \times [a]_{f^2}) \cap \beta = \emptyset$  and our claim holds. If  $3 \leq n_f(a) \neq \infty$ , then

$$([a]_{f^2} \times [a]_{f^2}) \setminus \beta = \{(x, y) \mid x, y \in [a]_{f^2} \text{ and } f^t(x) = f^t(y) \text{ for some } t \geq 0\},$$

which is obviously reflexive, symmetric and transitive. It is straightforward to see that  $\{a, f^2(a), \dots, f^{2n_{f^2}(a)}(a)\}$  is a complete irredundant system of representatives with respect to the equivalence relation  $([a]_{f^2} \times [a]_{f^2}) \sim \beta$ .  $\square$

A *partially anti-ordered unary algebra* is a triple  $(A, f, \leq_r)$ , where  $r$  is a partial order on  $A$  and  $f : A \rightarrow A$  is an order reversing map with respect  $r$ . Important facts about the close relationship between the cycle and the order structure in such (and similar) triples can be found in [FSz],[JPR],[L] and [Sz1].

**3.3. Proposition** (see [Sz1]). *If  $(A, f, \leq_r)$  is a partially anti-ordered unary algebra and  $(x, y) \in \pi$  is an  $f$ -incomparable pair, then  $x$  and  $y$  are incomparable with respect to  $r$ .*

**3.4. Corollary** (see [Sz1]). *If  $(A, f, \leq_r)$  is a partially anti-ordered unary algebra,  $r \subseteq R$  is an AMP extension of  $r$  and  $(x, y) \in \pi$  is an  $f$ -incomparable pair, then  $x$  and  $y$  are incomparable with respect to  $R$ .*

An AMP extension  $R$  of  $r$  is called  *$f$ -maximal*, if  $x \leq_R y$  or  $y \leq_R x$  for all  $f$ -comparable pairs  $(x, y) \in A \times A$  (that is  $(A \times A) \setminus \pi \subseteq R \cup R^{-1}$ ). Corollary 3.4 implies that any  $f$ -maximal AMP extension of  $r$  is maximal with respect to containment. We shall make use of the following notations:

$$\mathcal{M}(A, f, \leq_r) = \{R \mid r \subseteq R \text{ and } R \text{ is } f\text{-maximal AMP extension of } r\},$$

$$\mathcal{L}(A, f, \leq_r) = \{R \mid r \subseteq R \text{ and } R \text{ is an AMP linear order}\}.$$

Obviously,  $\mathcal{L}(A, f, \leq_r) \subseteq \mathcal{M}(A, f, \leq_r)$ . If the function  $f^2$  has a proper cycle or  $f$  has more than one fixed point, then  $\pi \neq \emptyset$  and Corollary 3.4 gives that  $\mathcal{L}(A, f, \leq_r) = \emptyset$ . If  $f^2$  has no proper cycle and  $f$  has at most one fixed point, then  $\pi = \alpha \cup \beta = \emptyset$  implies that  $\mathcal{L}(A, f, \leq_r) = \mathcal{M}(A, f, \leq_r)$ .

**3.5. Theorem** (see [L]). *If  $f^2$  has no proper cycle and  $f$  has at most one fixed point, then  $\mathcal{L}(A, f, \leq_r) \neq \emptyset$ .*

**3.6. Theorem** (see [Sz1]). *If  $(A, f, \leq_r)$  is an arbitrary partially anti-ordered unary algebra, then  $\mathcal{M}(A, f, \leq_r) \neq \emptyset$  and the elements of  $\mathcal{M}(A, f, \leq_r)$  are exactly the maximal (with respect to containment) elements in the set of all AMP extensions of  $r$ .*

**3.7. Theorem.** *Let  $f : A \rightarrow A$  be a function such that  $f^2$  has no proper cycles and  $f$  has at most one fixed point. Then there exists a distributive lattice  $(A, \vee, \wedge)$  on  $A$  such that  $f$  is a lattice anti-endomorphism of  $(A, \vee, \wedge)$ .*

**Proof.** A linearly ordered set is a distributive lattice and an order reversing map with respect to this linear order is a lattice anti-endomorphism. Thus the existence of the desired distributive lattice (chain) is an immediate consequence of Lengvárszky's Theorem 3.5.  $\square$

**3.8. Theorem.** *Let  $(A, f, \leq_r)$  be a partially anti-ordered unary algebra such that  $f$  has a cycle  $\{p, q\}$  of length 2. If  $x$  and  $y$  are  $r$ -incomparable for all  $x, y \in A$  with  $[x]_{f^2} \neq [y]_{f^2}$  and  $2 \neq n_f(x) \neq \infty$ , then there exists an extension  $R$  of  $r$  such that  $(A, \leq_R)$  is a lattice and  $f$  is a lattice anti-endomorphism of  $(A, \leq_R)$ .*

**Proof.** Let

$$A_0 = \{x \in A : 2 \neq n_f(x) \neq \infty\}$$

and

$$A_* = A \setminus A_0 = \{x \in A : n_f(x) = 2 \text{ or } n_f(x) = \infty\}.$$

We have either  $[x]_{f^2} \subseteq [x]_f \subseteq A_0$  or  $[x]_{f^2} \subseteq [x]_f \subseteq A_*$  for all  $x \in A$ . Clearly, both  $A_0$  and  $A_*$  are closed with respect to the action of  $f$ , i.e.  $f(A_0) \subseteq A_0$  and  $f(A_*) \subseteq A_*$ .

Take an arbitrary  $f$ -maximal AMP extension  $R$  of  $r$  (Theorem 3.6 ensures the existence of such  $R$ ). Since  $\pi \cap (A_* \times A_*) = \emptyset$  implies

$$A_* \times A_* \subseteq (A \times A) \setminus \pi \subseteq R \cup R^{-1},$$

we deduce that  $R_* = R \cap (A_* \times A_*)$  is a linear order on  $A_*$ . In view of  $p, q \in A_*$ , we may assume  $p \leq_{R_*} q$ .

For an appropriate subset  $\{x_t : t \in T\}$  of  $A_0$ , where the indices are taken from an index set  $T$ , we have  $\{[x]_{f^2} : x \in A_0\} = \{[x_t]_{f^2} : t \in T\}$ , and  $[x_t]_{f^2} \neq [x_s]_{f^2}$  for all  $t, s \in T$  with  $t \neq s$ . Such a subset  $\{x_t : t \in T\} \subseteq A_0$  is an *irredundant set of representatives* of the equivalence classes of  $\sim_{f^2}$  (in  $A_0$ ). That is

$$A_0 = \bigcup_{t \in T} [x_t]_{f^2} \text{ and } [x_t]_{f^2} \cap [x_s]_{f^2} = \emptyset \text{ for all } t, s \in T \text{ with } t \neq s.$$

Now for each  $t \in T$  consider the extension  $R_t = R \cap ([x_t]_{f^2} \times [x_t]_{f^2})$  of the restricted partial order  $r \cap ([x_t]_{f^2} \times [x_t]_{f^2})$ . By Corollary 3.2, the relation

$$R_t \cup R_t^{-1} = ([x_t]_{f^2} \times [x_t]_{f^2}) \setminus \pi = ([x_t]_{f^2} \times [x_t]_{f^2}) \setminus \beta$$

of  $R_t$ -comparability is an equivalence. Thus  $[x_t]_{f^2}$  is a disjoint union of finitely many  $R_t$ -chains (these are the equivalence classes with respect to  $([x_t]_{f^2} \times [x_t]_{f^2}) \setminus \beta$ ) and any two elements from different chains are incomparable with respect to  $R_t$ .

We claim that

$$S = R_* \cup \left( \bigcup_{t \in T} R_t \right) \cup P \cup Q$$

with

$$P = \{(a, x) : a \in A_*, x \in A_0 \text{ and } a \leq_{R_*} p\}$$

and

$$Q = \{(y, b) : b \in A_*, y \in A_0 \text{ and } q \leq_{R_*} b\}$$

is a lattice order extension of  $r$  and that  $f$  is a lattice anti-endomorphism of  $(A, \leq_S)$ .

The proof consists of the following straightforward steps.

Notice that,  $P \subseteq A_* \times A_0$ , and  $Q \subseteq A_0 \times A_*$ . Also the direct products  $A_* \times A_*$ ,  $A_* \times A_0$ ,  $A_0 \times A_*$ , and  $[x_t]_{f^2} \times [x_t]_{f^2}$  (for  $t \in T$ ) are pairwise disjoint.

In order to see  $r \subseteq S$ , take  $(u, v) \in r$ .

(1) If  $(u, v) \in A_* \times A_*$ , then  $r \cap (A_* \times A_*) \subseteq R \cap (A_* \times A_*) = R_* \subseteq S$  implies  $(u, v) \in R$ .

(2) If  $(u, v) \in A_* \times A_0$ , then  $[u]_{f^2} \neq [v]_{f^2}$  and  $2 \neq n_f(v) \neq \infty$  contradicts  $(u, v) \in r$ .

(3)  $(u, v) \in A_0 \times A_*$  is also impossible.

(4) If  $(u, v) \in A_0 \times A_0$ , then  $(u, v) \in [x_t]_{f^2} \times [x_s]_{f^2}$  for some  $t, s \in T$ . Clearly,  $t \neq s$  would imply  $[u]_{f^2} \neq [v]_{f^2}$ , and then  $2 \neq n_f(v) \neq \infty$  contradicts  $(u, v) \in r$ . Thus  $t = s$ , and  $r \cap ([x_t]_{f^2} \times [x_t]_{f^2}) \subseteq R \cap ([x_t]_{f^2} \times [x_t]_{f^2}) = R_t$  yields  $(u, v) \in S$ .

We prove that  $S$  is a partial order.

Antisymmetry: Let  $(u, v) \in S$  and  $(v, u) \in S$ .

(1) If  $(u, v), (v, u) \in R_*$ , then  $u = v$  follows from the antisymmetric property of  $R_*$

(2) If  $(u, v) \in R_t$  and  $(v, u) \in R_s$ , then  $t = s$ , and  $u = v$  follows from the antisymmetric property of  $R_t$ .

(3) If  $(u, v) \in P$  and  $(v, u) \in Q$ , then  $u \leq_{R_*} p$  and  $q \leq_{R_*} u$  imply  $q \leq_{R_*} p$ , contradicting with  $p \leq_{R_*} q$  and  $p \neq q$ .

(4) If  $(u, v) \in Q$  and  $(v, u) \in P$ , then interchanging the roles of  $u$  and  $v$  leads to a similar contradiction as in case (3).

Transitivity: Let  $(u, v) \in S$  and  $(v, w) \in S$ .

(1) If  $(u, v), (v, w) \in R_*$ , then  $(u, w) \in R_*$  follows from the transitivity of  $R_*$ .

(2) If  $(u, v) \in R_*$  and  $(v, w) \in P$ , then  $u \leq_{R_*} v \leq_{R_*} p$  and  $w \in A_0$  imply  $(u, w) \in P$ .

(3) If  $(u, v) \in R_t$  and  $(v, w) \in R_s$ , then we have  $t = s$ , and  $(u, w) \in R_t$  follows from the transitivity of  $R_t$ .

(4) If  $(u, v) \in R_t$  and  $(v, w) \in Q$ , then  $u, v \in A_0$ ,  $w \in A_*$ , and  $q \leq_{R_*} w$ . It follows that  $(u, w) \in Q$ .

(5) If  $(u, v) \in P$  and  $(v, w) \in R_t$ , then  $v, w \in A_0$ ,  $u \in A_*$ , and  $u \leq_{R_*} p$ . It follows that  $(u, w) \in P$ .

(6) If  $(u, v) \in P$  and  $(v, w) \in Q$ , then  $u \leq_{R_*} p \leq_{R_*} q \leq_{R_*} w$ , from which  $(u, w) \in R_*$  follows.

(7) If  $(u, v) \in Q$  and  $(v, w) \in R_*$ , then  $u \in A_0$  and  $q \leq_{R_*} v \leq_{R_*} w$  imply  $(u, w) \in P$ .

(8) If  $(u, v) \in Q$  and  $(v, w) \in P$ , then  $q \leq_{R_*} v \leq_{R_*} p$  contradicts  $p \leq_{R_*} q$  and  $p \neq q$ .

We note that  $f$  is order-reversing with respect to  $(A_*, \leq_{R_*})$ , and  $([x_t]_{f^2}, \leq_{R_t})$  for  $t \in T$ . In order to check the order-reversing property of  $f$  with respect to  $(A, \leq_S)$ , it is enough to see that  $(a, x) \in P$  implies  $(f(x), f(a)) \in Q$  and  $(y, b) \in Q$  implies  $(f(b), f(y)) \in P$ . Obviously,  $a \in A_*$ ,  $x \in A_0$ , and  $a \leq_{R_*} p$  imply  $f(a) \in A_*$ ,  $f(x) \in A_0$ , and  $q = f(p) \leq_{R_*} f(a)$ . Similarly,  $b \in A_*$ ,  $y \in A_0$ , and  $q \leq_{R_*} b$  imply  $f(b) \in A_*$ ,  $f(y) \in A_0$ , and  $f(b) \leq_{R_*} f(q) = p$ .

If  $u, v \in A$  are comparable elements with respect to  $S$ , then the existence of the supremum  $u \vee v$  and the infimum  $u \wedge v$  in  $(A, \leq_S)$  is evident; moreover, the order-reversing property of  $f$  ensures that

$$f(u \vee v) = f(u) \wedge f(v) \text{ and } f(u \wedge v) = f(u) \vee f(v).$$

If  $u, v \in A$  are incomparable elements with respect to  $S$ , then we have the following possibilities.

(1) If  $u \in A_*$  and  $v \in A_0$ , then  $(u, v) \notin P$ ,  $(v, u) \notin Q$ , and the linearity of  $R_*$  imply  $p \leq_{R_*} u \leq_{R_*} q$ , from which  $u \vee v = q$  and  $u \wedge v = p$  follow in  $(A, \leq_S)$ . Since  $f(u) \in A_*$ ,  $f(v) \in A_0$ , and  $p = f(q) \leq_{R_*} f(u) \leq_{R_*} f(p) = q$ , we deduce that

$$f(u \vee v) = f(q) = p = f(u) \wedge f(v) \text{ and } f(u \wedge v) = f(p) = q = f(u) \vee f(v).$$

(2) If  $u \in A_0$  and  $v \in A_*$ , then interchanging the roles of  $u$  and  $v$  leads to the same result as in case (1).

(3) If  $u, v \in A_0$  and  $[u]_{f^2} \neq [v]_{f^2}$ , then  $u \vee v = q$  and  $u \wedge v = p$  in  $(A, \leq_S)$  follow directly from the definition of  $S$ . Since  $f(u), f(v) \in A_0$  and  $[u]_{f^2} \neq [v]_{f^2}$  implies  $[f(u)]_{f^2} \neq [f(v)]_{f^2}$ , we deduce

$$f(u \vee v) = f(q) = p = f(u) \wedge f(v) \text{ and } f(u \wedge v) = f(p) = q = f(u) \vee f(v).$$

(4) If  $u, v \in A_0$  and  $[u]_{f^2} = [v]_{f^2} = [x_t]_{f^2}$  for some unique  $t \in T$ , then  $(u, v) \notin R_t$  and  $(v, u) \notin R_t$  imply  $(u, v) \in \beta$ . It follows that  $u$  and  $v$  are in different equivalence classes with respect to the  $R_t$ -comparability relation  $([x_t]_{f^2} \times [x_t]_{f^2}) \setminus \beta$ . An upper (lower) bound of  $\{u, v\}$  in  $([x_t]_{f^2}, \leq_{R_t})$  would be comparable with  $u$  and  $v$ , which



is impossible. We conclude that the set  $\{u, v\}$  has no upper and lower bounds in  $([x_t]_{f^2}, \leq_{R_t})$ . Thus we have  $u \vee v = q$  and  $u \wedge v = p$  in  $(A, \leq_S)$ .

Since  $f(u), f(v) \in [f(x_t)]_{f^2}$  and  $(f(u), f(v)) \in \beta$  (by Lemma 3.1), a similar argument (in  $([x_s]_{f^2}, \leq_{R_s})$  with  $[x_s]_{f^2} = [f(x_t)]_{f^2}$ ) gives that

$$f(u) \vee f(v) = q = f(p) = f(u \wedge v) \text{ and } f(u) \wedge f(v) = p = f(q) = f(u \vee v). \square$$

Theorems 3.7 and 3.8 together generalize the answer given in section 2 to the question in the title of the paper. We pose a further problem.

**Problem.** *Consider an arbitrary function  $f : A \rightarrow A$ . Find necessary and sufficient conditions for the existence of a modular (or distributive) lattice structure  $(A, \vee, \wedge)$  on  $A$  such that  $f$  becomes a lattice anti-endomorphism of  $(A, \vee, \wedge)$ .*

**Example.** Let  $A = \{p, q, x_1, x_2, \dots, x_n\}$ , where  $n = 2k + 1 \geq 3$  is odd, and let  $f : A \rightarrow A$  be a function with  $f(p) = q$ ,  $f(q) = p$ ,  $f(x_n) = x_1$ , and  $f(x_i) = x_{i+1}$  for  $1 \leq i \leq n - 1$ . If  $f$  is an anti-endomorphism of some lattice  $(A, \leq, \vee, \wedge)$ , then  $f$  is order-reversing with respect to  $(A, \leq)$ , and Proposition 3.3 ensures that the proper cycle  $\{x_1, \dots, x_n\}$  of  $f^2$  is an antichain in  $(A, \leq)$ . Since  $x_1 \vee \dots \vee x_n$  and  $x_1 \wedge \dots \wedge x_n$  form a two-element cycle of  $f$ , one of  $x_1 \vee \dots \vee x_n$  and  $x_1 \wedge \dots \wedge x_n$  is  $p$  and the other is  $q$ . Thus  $(A, \leq, \vee, \wedge)$  is isomorphic to the lattice  $M_n$  in both cases. It follows that there is no distributive lattice structure on  $A$  making  $f$  a lattice anti-endomorphism (even though  $f$  has a cycle of length 2).

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